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2002 J. Phys. A: Math. Gen. 35 3535

(http://iopscience.iop.org/0305-4470/35/15/312)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 3535–3546 PII: S0305-4470(02)32872-5

Laguerre moments and generalized functions

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Received 16 January 2002, in final form 15 February 2002
Published 5 April 2002
Online at stacks.iop.org/JPhysA/35/3535

Abstract
Here we explore the link between the moments of the Laguerre polynomials or *Laguerre moments* and the *generalized functions* (as the Dirac delta-function and its derivatives), presenting several interesting relations. A useful application is related to a procedure for calculating mean values in quantum optics that makes use of the so-called quasi-probabilities. One of them, the *P-distribution*, can be represented by a sum over Laguerre moments when the electromagnetic field is in a photon-number state. Consequently, the P-distribution can be expressed in terms of Dirac delta-function and derivatives. More specifically, we found a direct relation between P-distributions and the *Laguerre factorial moments*.

PACS numbers: 02.30.Gp, 02.30.Lt, 42.50.−p

1. Introduction

The probability finding, as a measurement outcome, n photons in the field state $\hat{\rho}$ is given by $\mathrm{Tr}\,(\hat{\rho}|n\rangle\langle n|)$ ($\hat{\rho}$ is a traceclass operator and $|n\rangle$ is the eigenstate of the photon-number operator). So, in a field prepared in a coherent state $|\alpha\rangle$ ($\hat{\rho} = |\alpha\rangle\langle\alpha|$), where α is a complex number and $|\alpha|^2$ is the intensity of the field, or the mean photon number, then $\mathrm{Tr}(\hat{\rho}|n\rangle\langle n|) = |\langle n|\alpha\rangle|^2$.

On the other hand, from a formal point of view the state $|\alpha\rangle$ is used to map a q-number operator $\hat{O}(a, a^\dagger)$ (a and $a^\dagger$ are destruction and creation operators of photons with respect to the number state $|n\rangle$, $n = 0, 1, 2, \ldots$) to a c-number function. The trace operation $\mathrm{Tr}(\hat{\rho}|\alpha\rangle\langle\alpha|) = \langle\alpha|\hat{\rho}|\alpha\rangle = Q_{\hat{\rho}}(\alpha, \alpha^*)$ defines the Husimi distribution, or Q-distribution, for state $\hat{\rho}$ (actually this is a map: $\hat{\rho} \Rightarrow Q_{\hat{\rho}}(\alpha, \alpha^*)$, $a \to \alpha$, $a^\dagger \to \alpha^*$).

The mean value of an operator $\hat{O}(a, a^\dagger)$ can be written as

$$\mathrm{Tr}(\hat{\rho}\hat{O}) = \int O(\alpha, \alpha^*) P_{\hat{\rho}}(\alpha, \alpha^*)\, \mathrm{d}^2\alpha \qquad (1)$$

where $O(\alpha,\alpha^*) = \langle\alpha|\hat{O}|\alpha\rangle$ (this is also a map: $\hat{O} \Rightarrow O(\alpha,\alpha^*)$) and $P_{\hat\rho}(\alpha,\alpha^*)$ is the Glauber–Sudarshan or P-distribution, related to $Q_{\hat\rho}(\alpha,\alpha^*)$ through

$$P_{\hat\rho}(\alpha,\alpha^*) = \exp\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) Q_{\hat\rho}(\alpha,\alpha^*).$$

The distributions $Q_{\hat\rho}(\alpha,\alpha^*)$ and $P_{\hat\rho}(\alpha,\alpha^*)$ are quasi-probabilities, the former is always a smooth and well-behaved function of its arguments while the latter, depending on the state $\hat\rho$, may be a regular function or a *generalized function* (GF) as is the case for $\hat\rho = |n\rangle\langle n|$.

For a field in state $\hat\rho = |n\rangle\langle n|$ the probability to find n photons in a coherent state $|\alpha\rangle\langle\alpha|$ is the same as the Q-distribution for state $|n\rangle\langle n|$, being a Poisson distribution in variable n with mean value $|\alpha|^2$ [1],

$$Q_n(|\alpha|^2) = |\langle n|\alpha\rangle|^2 = \frac{\exp(-|\alpha|^2)|\alpha|^{2n}}{n!} \tag{2}$$

a smooth and well-behaved function of its argument. However, the P-distribution is a quite singular function, as was originally reported in the classical papers of Glauber [2] and Sudarshan [3] and more recently reviewed by Wünsche [4], who found new relations and representations for the P-distribution. Working on this same problem of representing the P-distribution, we obtained several results which we did not find in the current literature, relating the Laguerre moments and Laguerre factorial moments to GFs. We also derived a direct relation between the P-distribution and the Laguerre factorial moments. These results are reported in this paper.

We begin by reminding, with some examples, how the Dirac delta-function arises in mathematical physics [5–10]:

(I) Certain sequences of functions defined on $\mathbb{R}$, $f_n(x)$, $n = 1, 2, 3, \ldots$ are well behaved (continuous with continuous derivatives to all orders) in a domain $\mathcal{I}$; however, they cease to exist as such when $n \to \infty$, acquiring meaning as a continuous linear functional $T_f\phi \equiv \langle f,\phi\rangle = \int_{\mathcal{I}} f(x)\phi(x)\,\mathrm{d}x$ that maps each continuous test function $\phi(x)$ ($\phi \in \Phi$, Φ is a linear vector space) onto a complex number. So, the functional denoted as T_f (or simply f) is called the distribution or GF. For instance, the functions

$$\frac{n}{\sqrt{\pi}}\,\mathrm{e}^{-n^2/x^2} \qquad \frac{n}{\pi}\frac{1}{1+n^2x^2} \qquad \frac{\sin nx}{\pi x} \tag{3}$$

although being continuous with continuous derivatives to all orders for any finite integer n, no longer exhibit this property in the limit $n \to \infty$, thus no longer belong to the class of regular functions. All the examples in (3) converge to the so-called Dirac delta-function δ, in reality a GF, to be referred to as the Dirac distribution (DD)

$$\delta(x) = \lim_{n\to\infty}\frac{n}{\sqrt{\pi}}\,\mathrm{e}^{-n^2/x^2} = \lim_{n\to\infty}\frac{n}{\pi}\frac{1}{1+n^2x^2} = \lim_{n\to\infty}\frac{\sin nx}{\pi x}$$

defined by the functional

$$\langle T_{\delta_a},\phi\rangle = \lim_{n\to\infty}\int_{\mathcal{I}}\delta_n(x-x_0)\phi(x)\,\mathrm{d}x = \phi(x_0)$$

where $\delta_n(x)$ stands for any one of the functions displayed in (3) and $\phi(x)$ is a test function.

(II) From Sturm–Liouville theory we know that a class of second-order differential equations accept, as solution, orthogonal polynomials $\mathcal{P}_n(x)$ that form a complete set, meaning that any piecewise smooth and bounded function $f(x)$ defined on $\mathcal{I}$ ($x \in \mathcal{I}$) can be expanded in terms of the $\mathcal{P}_n(x)$ (the weight function and normalization factors are included in it),

$$f(x) = \sum_n c_n\mathcal{P}_n(x)$$

the coefficients are obtained by integration,

$$c_n = \int_{\mathcal{I}} f(x)\mathcal{P}_n(x)\,\mathrm{d}x$$

and

$$\sum_{n=0}^{\infty} \mathcal{P}_n(x)\mathcal{P}_n(x') = \delta(x - x')$$

is the completeness property of the polynomials. If the point 0 is contained in $\mathcal{I}$, then setting $x' = 0$ in the previous equation, the DD becomes equal to an infinite weighted sum of polynomials,

$$\sum_{n=0}^{\infty} \mathcal{P}_n(0)\mathcal{P}_n(x) = \delta(x).$$

Concerning the weighted Laguerre polynomials, $\mathcal{P}_n(x) = \mathrm{e}^{-x/2}L_n(x)$, which are defined on $[0, \infty)$ with $\mathcal{P}_n(0) = 1$, one notes that the infinite sum expansion

$$\sum_{n=0}^{\infty} L_n(x) = \delta_+(x) \tag{4}$$

is a representation of a GF (we will come back to this point in the next section, with a proper demonstration)[3]. The GF on the right-hand side (RHS) of equation (4) is related to the Dirac distribution

$$\int_0^{\infty} \delta_+(x)\phi(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0^+} \int_0^{\infty} \delta(x-\varepsilon)\phi(x)\,\mathrm{d}x = \phi(0) \tag{5}$$

since GFs are properly defined in open intervals.

The infinite sums of polynomials and moments are useful for a certain class of problems, as we will see in section 4. In what follows, we shall consider the associated Laguerre polynomials $L_n^{\alpha}(x)$, whose generating function (GEF) is

$$G(x, t, \alpha) = \mathrm{e}^{-\frac{xt}{1-t}}\Big/(1-t)^{\alpha+1} \tag{6}$$

since

$$G(x, t, \alpha) = \sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n \qquad |t| < 1 \tag{7}$$

where t is a complex variable in the open disc of radius $|t|$, $t \in \mathcal{D}_t = \{t \in \mathbb{C} \mid 0 \leqslant |t| < 1\}$.

In section 2 we present some lemmas involving the ordinary Laguerre polynomials and the theorem for the Laguerre moments, whereas in section 3 we extend the results to the associated Laguerre polynomials. In section 4 we make use of previous results and obtain an expression for the P-distribution in terms of either the Laguerre factorial moments or the GFs. In section 5 we expose our conclusions.

2. The ordinary Laguerre polynomials and moments

Initially, we shall consider the ordinary Laguerre polynomials ($\alpha = 0$), $L_n^0(x) \equiv L_n(x)$, where $L_n(0) = 1$. If we extend the domain of t to include the additional point $t = +1$ in

[3] Here the $\delta_+(x)$ should not be confused with the distributions $\delta^{\pm} = \frac{\delta}{2} \pm \frac{1}{2\pi i} vp\frac{1}{x}$ defined in [6], p 91, where vp stands for the Cauchy principal value.

equations (6) and (7) then $\mathcal{D}'_t = \{\mathcal{D}_t, 1\}$; one verifies that at this point (since $t = |t|\,\mathrm{e}^{\mathrm{i}\varphi}$, $|t| = 1$ and $\varphi = 0$) the GEF (6) becomes

$$\lim_{t\to 1^-} G(x,t,0) = \lim_{\varepsilon\to 0^+} \frac{\mathrm{e}^{-\frac{x}{\varepsilon}}}{\varepsilon} = \begin{cases} 0 & \text{for} \quad x > 0 \\ \infty & \text{for} \quad x = 0 \end{cases}. \tag{8}$$

Thus $G(x, 1, 0)$ is no longer a regular function in the usual sense, it becomes quite singular at $x = 0$. Let us look more closely at GEF (6) and analyse its properties:

Lemma 1. *For $x \in \mathbb{R}_+$, $\mathbb{R}_+ \equiv (0,\infty)$, the $\alpha = 0$ GEF $G(x, 1, 0)$ can be represented by the GF, equation (5),*

$$G(x,1,0) = \delta_+(x). \tag{9}$$

Proof. Multiplying $G(x, t, 0)$ by a piecewise smooth test function $\phi(x)$, with $\phi \in \mathbb{R}$ and $x \in \mathbb{R}_+$, integrating

$$\int_0^\infty \frac{\mathrm{e}^{-\frac{xt}{1-t}}}{(1-t)}\phi(x)\,\mathrm{d}x$$

performing the change of variable $x = y(1-t)/t$ and considering the limit $t \to 1^-$, we get

$$\lim_{t\to 1^-}\int_0^\infty \mathrm{e}^{-y}\frac{1}{t}\phi\left(\frac{1-t}{t}y\right)\mathrm{d}y = \phi(0)$$

which is the main property of the GF, thus

$$\lim_{\varepsilon\to 0^+}\int_0^\infty G(x, 1-\varepsilon, 0)\phi(x)\,\mathrm{d}x = \phi(0) \tag{10}$$

and equation (9) is justified, i.e., $\lim_{t\to 1^-} G(x,t,0)$ is a representation of the GF. □

Symbolically, equation (10) can be written as

$$G(x,1,0)\phi(x) = \phi(0) \quad \text{or} \quad \delta_+(x)\phi(x) = \phi(0)$$

which is a well-known property of the DD, omnipresent in mathematical physics textbooks [6, 9, 10].

As the summation $\sum_{n=0}^{N} L_n(x)$ is a regular function for any finite N, we may ask: does the infinite summation go to a GF? Or, is the equality $\sum_{n=0}^{\infty} L_n(x) = \delta_+(x)$ true? Before answering that question we first recall the following theorem of the Laguerre polynomials (we do not present the demonstration since it can be found in the usual textbooks [11]):

Theorem 1. *If the real function $f(x)$, defined in the interval $[0,\infty)$, is piecewise smooth in every subinterval $[x_1, x_2]$, where $0 \leqslant x_1 < x_2 < \infty$ and if the integral*

$$\int_0^\infty \mathrm{e}^{-x}x^\alpha\,[f(x)]^2\,\mathrm{d}x$$

is finite, then the series

$$\sum_{n=0}^{\infty} c_{n,\alpha}L_n^\alpha(x)$$

with coefficients

$$c_{n,\alpha} = \frac{n!}{\Gamma(n+\alpha+1)}\int_0^\infty \mathrm{e}^{-x}x^\alpha f(x)L_n^\alpha(x)\,\mathrm{d}x$$

converges to $f(x)$ *at every continuity point of* $f(x)$. *At a discontinuity point* x_0 *the series converges to*

$$\frac{1}{2}\lim_{\varepsilon\to 0}[f(x_0+\varepsilon)+f(x_0-\varepsilon)].$$

For $\alpha = 0$ we have

$$f(x) = \sum_{n=0}^{\infty} c_n L_n(x) \tag{11}$$

with

$$c_n = \int_0^{\infty} \mathrm{e}^{-x} f(x) L_n(x)\,\mathrm{d}x. \tag{12}$$

The Laguerre polynomials are defined such that $L_n(0) = 1$, implying that $f(0) = \sum_{n=0}^{\infty} c_n$; so we propose

Lemma 2. *From equation (12) and for* $x \in \mathbb{R}_+ \equiv (0, \infty)$ *we obtain*

$$\sum_{n=0}^{\infty} L_n(x) = \delta_+(x). \tag{13}$$

Proof. Summing over all n on both sides of equation (12) and interchanging the order of summation and integration, we get

$$\sum_{n=0}^{\infty} c_n = f(0) = \int_0^{\infty} \mathrm{e}^{-x} f(x) \left(\sum_{n=0}^{\infty} L_n(x)\right) \mathrm{d}x$$

so we verify equation (13). □

We can also verify equation (13) by using a recurrence relation of the Laguerre polynomials and a property of the DD:

Corollary 1. *From the recurrence relation of the Laguerre polynomials*

$$x\mathrm{d}_x L_n(x) = nL_n(x) - nL_{n-1}(x) \qquad n = 1, 2, 3, \ldots \tag{14}$$

where $\mathrm{d}_x \doteq \mathrm{d}/\mathrm{d}x$, *follows the known relation of the DD*

$$x\mathrm{d}_x\delta_+(x) = -\delta_+(x) \tag{15}$$

for $\sum_{n=0}^{\infty} L_n(x) = \delta_+(x)$.

Proof. Summing both sides of equation (14) over n, from 1 to N, we get

$$\sum_{n=1}^{N} nL_n(x) = x\mathrm{d}_x \sum_{n=1}^{N} L_n(x) + \sum_{n=1}^{N} nL_{n-1}(x)$$

or

$$\sum_{n=0}^{N} nL_n(x) = x\mathrm{d}_x \sum_{n=0}^{N} L_n(x) + \sum_{n=0}^{N} (n+1)L_n(x)$$

then

$$x\mathrm{d}_x \left(\sum_{n=0}^{N} L_n(x)\right) = -\sum_{n=0}^{N} L_n(x).$$

Considering $N \to \infty$ we recognize equation (15) for $\sum_{n=0}^{\infty} L_n(x) = \delta_+(x)$. □

Therefore, we can write

$$G(x,1,0)=\sum_{n=0}^{\infty}L_n(x)=\delta_+(x). \tag{16}$$

We now give some relations involving the GFs that will be necessary to demonstrate a useful theorem. As a preliminary, we

(i) write in short $\delta_+^{(n)}(x)\doteq(\mathrm{d}_x)^n\delta_+(x)$ (with $\delta_+^{(0)}(x)\doteq\delta_+(x)$);
(ii) assume $\lim_{x\to0}\left(x\sum_{n=0}^{\infty}L_n(x)\right)\doteq0$ and $\lim_{x\to0}\left(x\,(\mathrm{d}_x)^n\sum_{m=0}^{\infty}L_m(x)\right)\doteq0$, thus

$$\lim_{x\to0}x\delta_+^{(n)}(x)=\lim_{x\to0}x\,(\mathrm{d}_x)^n\,G(x,1,0)=0; \tag{17}$$

(iii) introduce the bracketed terms $\left[x\delta_+^{(1)}(x)\right]$, $\left[x\delta_+^{(2)}(x)\right],\ldots,\left[x\delta_+^{(n)}(x)\right]$ as GFs;
(iv) define the functional

$$\left(\left[x\delta_+^{(n)}(x)\right],\phi\right)\doteq\int_0^\infty\left[x\delta_+^{(n)}(x)\right]\phi(x)\,\mathrm{d}x$$

where $\phi(x)$ is a regular piecewise and bounded test function in $\mathbb{R}_+$.

Example 1. For $n=1$, the functional is

$$\int_0^\infty\left[x\delta_+^{(1)}(x)\right]\phi(x)\,\mathrm{d}x=\int_0^\infty\delta_+^{(1)}(x)(\phi(x)x)\,\mathrm{d}x=-\int_0^\infty\delta_+(x)[\,\mathrm{d}_x(\phi(x)x)]\,\mathrm{d}x$$
$$=-\int_0^\infty\delta_+(x)(x\mathrm{d}_x\phi(x)+\phi(x))\,\mathrm{d}x=\int_0^\infty(-\delta_+(x))\phi(x)\,\mathrm{d}x=-\phi(0).$$

or in symbolic notation

$$\left[x\delta_+^{(1)}(x)\right]=-\delta_+(x) \tag{18}$$

the term in the brackets is reduced to the GF multiplied by -1.

Example 2. For $n=2$,

$$\int_0^\infty\left[x\delta_+^{(2)}(x)\right]\phi(x)\,\mathrm{d}x=\int_0^\infty\delta_+^{(2)}(x)\,(\phi(x)x)\,\mathrm{d}x=(-1)^2\int_0^\infty\delta_+(x)\left[(\mathrm{d}_x)^2\,(\phi(x)x)\right]\mathrm{d}x$$
$$=\int_0^\infty\delta_+(x)\left[x(\mathrm{d}_x)^2\phi(x)+2\mathrm{d}_x\phi(x)\right]\mathrm{d}x=2\int_0^\infty\delta_+(x)\,[\mathrm{d}_x\phi(x)]\,\mathrm{d}x$$
$$=\int_0^\infty\left(-2\delta_+^{(1)}(x)\right)\phi(x)\,\mathrm{d}x=2\phi'(0)$$

therefore, in symbolic notation

$$\left[x\delta_+^{(2)}(x)\right]=-2\delta_+^{(1)}(x). \tag{19}$$

Remark 1. The bracket $[x\delta_+(x)]=0$, since

$$\int_0^\infty[x\delta_+(x)]\phi(x)\,\mathrm{d}x=\int_0^\infty\delta_+(x)[\phi(x)x]\,\mathrm{d}x=0.$$

The general term $\left[x\delta_+^{(n)}(x)\right]$ is obtained by induction:

Lemma 3. *For $x\in\mathbb{R}_+$, the factor $(-x)$ in $\left[(-x)\delta_+^{(n)}(x)\right]$ acts as a first-order derivative on $\delta_+^{(n)}(x)$,*

$$\left[x\delta_+^{(n)}(x)\right]=-n\delta_+^{(n-1)}(x)\qquad n=1,2,3,\ldots. \tag{20}$$

Proof. From examples 1 and 2 we have $\left[x\delta_+^{(1)}(x)\right] = -\delta_+(x)$ and $\left[x\delta_+^{(2)}(x)\right] = -2\delta_+^{(1)}(x)$. For $\left[x\delta_+^{(3)}(x)\right]$ we use this last equation,

$$\left[x\delta_+^{(3)}(x)\right] = \mathrm{d}_x\left[x\delta_+^{(2)}(x)\right] - \delta_+^{(2)}(x) = \mathrm{d}_x\left(-2\delta_+^{(1)}(x)\right) - \delta_+^{(2)}(x) = -3\delta_+^{(2)}(x) \tag{21}$$

and so forth for higher order derivatives, so equation (20) stands for any positive integer n. □

We can generalize this result for higher powers of x through

Lemma 4. *For $x \in \mathbb{R}_+$, any positive integer p, and assuming relation (20), it follows that in $\left[(-x)^p\delta_+^{(n)}(x)\right]$ the factor $(-x)^p$ acts as a pth-order derivative multiplied by a constant,*

$$\left[x^p\delta_+^{(n)}(x)\right] = \begin{cases} (-1)^p\frac{n!}{(n-p)!}\delta_+^{(n-p)}(x) & \text{for} \quad n \geqslant p \\ 0 & \text{for} \quad n < p. \end{cases} \tag{22}$$

Proof. Since $\left[x\delta_+^{(n)}(x)\right] = -n\delta_+^{(n-1)}(x)$ then

$$\begin{aligned}\left[x^2\delta_+^{(n)}(x)\right] &= \left[x\left(-n\delta_+^{(n-1)}(x)\right)\right] \\ &= -n\left[x\delta_+^{(n-1)}(x)\right] = -n\,(-n+1)\,\delta_+^{(n-2)}(x) \qquad \text{for} \quad n \geqslant 2.\end{aligned}$$

However, $\left[x^2\delta_+^{(1)}(x)\right] = -\left[x\delta_+(x)\right] = 0$, where the second equality follows from remark 1. Repeating this procedure for any positive integer p, we verify equation (22). □

Lemma 5. *For the differential operator*

$$\Lambda(x) \doteq -\left((1-x)\mathrm{d}_x + xd_x^2\right)$$

the following equation

$$\left[\Lambda(x)\delta_+^{(n)}(x)\right] = (n+1)\left(\delta_+^{(n+1)}(x) - \delta_+^{(n)}(x)\right) \tag{23}$$

holds for $n = 0, 1, 2, \ldots$.

Proof. Setting $n = 0$ in equation (23) and by using relation (22) we get

$$\begin{aligned}\left[\Lambda(x)\delta_+(x)\right] &= -\left(\left[(1-x)\delta_+^{(1)}(x)\right] + \left[x\delta_+^{(2)}(x)\right]\right) \\ &= -\left(\delta_+^{(1)}(x) - (-\delta_+(x)) + \left(-\frac{2!}{1!}\delta_+^{(1)}(x)\right)\right) \\ &= \delta_+^{(1)}(x) - \delta_+(x).\end{aligned} \tag{24}$$

Following the same procedure for $n \geqslant 1$ we get equation (23). □

Lemma 6. *For any positive integer s we have*

$$\begin{aligned}\left[(\Lambda(x))^s\,\delta_+^{(r)}(x)\right] &= \left[\prod_{k=1}^{s}(r+k)\right]\delta_+^{(s+r)}(x) - \left[\prod_{k=1}^{s-1}(r+k)\sum_{j=1}^{s}(r+j)\right]\delta_+^{(s+r-1)}(x) \\ &\quad + \left[\prod_{k=1}^{s-2}(r+k)\sum_{j=1}^{s-1}(r+j)\sum_{l=j}^{s-1}(r+l)\right]\delta_+^{(s+r-2)}(x) \\ &\quad - \left[\prod_{k=1}^{s-3}(r+k)\sum_{j=1}^{s-2}(r+j)\sum_{l=j}^{s-2}(r+l)\sum_{m=l}^{s-2}(r+m)\right]\delta_+^{(s+r-3)}(x) + \cdots \\ &\quad + \left[(-1)^s\underbrace{\sum_{j=1}^{1}(r+j)\sum_{l=j}^{1}(r+l)\cdots\sum_{n=m}^{1}(r+n)}_{\text{product of } s \text{ summation symbols}}\right]\delta_+^{(r)}(x).\end{aligned} \tag{25}$$

Proof. Starting with equation (23) and applying $\Lambda(x)$ successively, by induction we arrive at equation (25). □

We now define the Laguerre moments

$$\mathcal{M}_s(x,0) \doteq \sum_{n=0}^{\infty} n^s L_n(x) \tag{26}$$

and propose the following theorem.

Theorem 2. *For any positive integer s and $x \in \mathbb{R}_+$, the moments $\mathcal{M}_s(x,0)$ can be expanded in terms of GFs as*

$$\mathcal{M}_s(x,0) = \sum_{l=0}^{s} c_{s,l}\delta_+^{(l)}(x) \tag{27}$$

with the coefficients given by

$$c_{s,l} = (-1)^{s-l} l! \underbrace{\sum_{j=1}^{l+1} j \sum_{k=j}^{l+1} k \cdots \sum_{n=m}^{l+1} n}_{\text{product of } s-l \text{ summation symbols}}. \tag{28}$$

Proof. The eigenvalue equation for the Laguerre polynomials is

$$\Lambda(x)L_n(x) = nL_n(x). \tag{29}$$

Summing both sides of the equation over n from 0 to ∞ gives the first moment,

$$\begin{aligned}\mathcal{M}_1(x,0) &= \sum_{n=0}^{\infty} nL_n(x)\\ &= [\Lambda(x)\delta_+(x)] = \delta_+^{(1)}(x) - \delta_+(x)\end{aligned} \tag{30}$$

where we used equation (24) to obtain the second equality. By applying $s-1$ times $\Lambda(x)$ on both sides of equation (29) we obtain

$$(\Lambda(x))^s L_n(x) = n^s L_n(x)$$

summing over n we get the sth-order moment

$$\mathcal{M}_s(x,0) = \sum_{n=0}^{\infty} n^s L_n(x) = [(\Lambda(x))^s \delta_+(x)].$$

Setting $r = 0$ in equation (25) we get

$$\begin{aligned}\mathcal{M}_s(x,0) &= s!\delta_+^{(s)}(x) - (s-1)!\left(\sum_{j=1}^{s} j\right)\delta_+^{(s-1)}(x) + (s-2)!\left(\sum_{j=1}^{s-1} j \sum_{k=j}^{s-1} k\right)\delta_+^{(s-2)}(x)\\ &\quad - (s-3)!\left(\sum_{j=1}^{s-2} j \sum_{k=j}^{s-2} k \sum_{m=k}^{s-2} m\right)\delta_+^{(s-3)}(x) + \cdots + (-)^s\delta_+(x)\\ &= \sum_{l=0}^{s}\left[(-1)^{s-l} l! \underbrace{\sum_{j=1}^{l+1} j \sum_{k=j}^{l+1} k \cdots \sum_{n=m}^{l+1} n}_{\text{product of } s-l \text{ summation symbols}}\right]\delta_+^{(l)}(x) = \sum_{l=0}^{s} c_{s,l}\delta_+^{(l)}(x).\end{aligned}$$

□

Example 3.

$$\begin{aligned}\mathcal{M}_2(x,0) &= [\Lambda(x)\,[\Lambda(x)\delta_+(x)]] = \left[\Lambda(x)\left(\delta_+^{(1)}(x)-\delta_+(x)\right)\right]\\ &= 2\left(\delta_+^{(2)}(x)-\delta_+^{(1)}(x)\right)-\left(\delta_+^{(1)}(x)-\delta_+(x)\right)\\ &= 2\delta_+^{(2)}(x)-3\delta_+^{(1)}(x)+\delta_+(x)\end{aligned} \tag{31}$$

proceeding analogously we get

$$\mathcal{M}_3(x,0) = 6\delta_+^{(3)}(x)-12\delta_+^{(2)}(x)+7\delta_+^{(1)}(x)-\delta_+(x) \tag{32}$$

and

$$\mathcal{M}_4(x,0) = 24\delta_+^{(4)}(x)-60\delta_+^{(3)}(x)+50\delta_+^{(2)}(x)-15\delta_+^{(1)}(x)+\delta_+(x). \tag{33}$$

It can be verified that

$$\sum_{l=0}^{s} c_{s,l} = 0. \tag{34}$$

2.1. *The Laguerre factorial moments*

The Laguerre factorial moment of order s, associated with the ordinary Laguerre polynomials, is defined as

$$\mathcal{F}_s(x,0) \doteq \sum_{n=s}^{\infty} n(n-1)(n-2)\cdots(n-s+1)L_n(x) = \sum_{n=s}^{\infty}\frac{n!}{(n-s)!}L_n(x) \tag{35}$$

and it is related to the Laguerre moments through

$$\mathcal{F}_s(x,0) = \sum_{r=1}^{s} S_s^{(r)}\mathcal{M}_r(x,0) \tag{36}$$

where the coefficients $S_s^{(r)}$ are the Stirling numbers of the first kind [12][4]. By substituting equation (27) into (35) and after summing over the coefficients we arrive at a simple expression for the factorial moments in terms of ultradistributions,

$$\mathcal{F}_s(x,0) = s!\sum_{r=0}^{s}(-1)^{s-r}\binom{s}{r}\delta_+^{(r)}(x). \tag{37}$$

3. The associated Laguerre polynomial and moments

Lemma 7. *For α a positive integer and $x\in\mathbb{R}_+$, the equation*

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) = (1-\,\mathrm{d}_x)^{\alpha}\,\delta_+(x) \tag{38}$$

holds and for $t=1$ the GEF also becomes

$$G(x,1,\alpha) = \lim_{t\to 1}\frac{\mathrm{e}^{-\frac{xt}{1-t}}}{(1-t)^{\alpha+1}} = (1-\mathrm{d}_x)^{\alpha}\,\delta_+(x). \tag{39}$$

[4] The Stirling numbers of the first kind have the properties $S_s^{(0)}=\delta_{s,0}$ and $\sum_{r=1}^{s} S_s^{(r)}=0$. The inverse relation is $\mathcal{M}_r(x,0)=\sum_{l=0}^{r}\mathcal{S}_r^{(l)}\mathcal{F}_l(x,0)$ where $\mathcal{S}_r^{(l)}$ are the Stirling numbers of the second kind.

Proof. Considering the following properties of the Laguerre polynomials, [13, 14]

$L_n^\alpha(x) = (-\mathrm{d}_x)^\alpha L_{n+\alpha}(x)$ and $(\mathrm{d}_x)^\alpha L_{n+\alpha}(x) = (-1 + \mathrm{d}_x)^\alpha L_n(x)$

it follows that

$$L_n^\alpha(x) = (1 - \mathrm{d}_x)^\alpha L_n(x) \tag{40}$$

and summing both sides of equation (40) over n from 0 to ∞ one gets equation (38).

Concerning equation (39), looking at $G(x, t, \alpha)$, equation (7), one can easily verify that

$$G(x, t, \alpha) = (1 - \partial_x)^\alpha G(x, t, 0) \tag{41}$$

($\partial_x \equiv \partial/\partial x$) thus

$$\lim_{t\to 1^-} G(x, t, \alpha) = (1 - \mathrm{d}_x)^\alpha \lim_{t\to 1^-} G(x, t, 0) = (1 - \mathrm{d}_x)^\alpha \delta_+(x)$$

(the second equality follows from lemma 1) so verifying equation (39). Therefore, one arrives at

$$G(x, 1, \alpha) = \sum_{n=0}^{\infty} L_n^\alpha(x) = (1 - \mathrm{d}_x)^\alpha \delta_+(x) \tag{42}$$

which generalizes equation (9). □

We now propose the following theorem.

Theorem 3. *For any positive integers α and s, and $x \in \mathbb{R}_+$, the moments $\mathcal{M}_s(x;\alpha)$ are given by the following summation of GFs:*

$$\mathcal{M}_s(x;\alpha) = \sum_{n=0}^{\infty} n^s L_n^\alpha(x) = \sum_{l=0}^{s} c_{s,l} \sum_{j=0}^{\alpha} (-)^j \binom{\alpha}{j} \delta_+^{(l+j)}(x). \tag{43}$$

Proof. Considering equation (40), multiplying both sides by n^s (s a non-negative integer) and summing over n from 0 to ∞, one gets

$$\mathcal{M}_s(x;\alpha) = (1 - \mathrm{d}_x)^\alpha \mathcal{M}_s(x;0) = (1 - \mathrm{d}_x)^\alpha \sum_{l=0}^{s} c_{s,l}\delta_+^{(l)}(x)$$

$$= \sum_{l=0}^{s} c_{s,l} \sum_{j=0}^{\alpha} (-)^j \binom{\alpha}{j} \delta_+^{(l+j)}(x). \tag{44}$$

□

3.1. The factorial moments

The associated Laguerre factorial moment of order s is defined as

$$\mathcal{F}_s(x,\alpha) \doteq \sum_{n=s}^{\infty} \frac{n!}{(n-s)!} L_n^\alpha(x) \tag{45}$$

and using relations (44) we get

$$\mathcal{F}_s(x,\alpha) = (1 - \mathrm{d}_x)^\alpha \mathcal{F}_s(x, 0) \tag{46}$$

$$= s! \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \delta_+^{(r+j)}(x). \tag{47}$$

4. The P-distribution for the photon-number state

Writing the complex variable α in the polar form and calling $|\alpha|^2 = y$ in equation (2) we obtain

$$Q_n(y) = \frac{\exp(-y)y^n}{n!}$$

and

$$P_n(y) = \frac{1}{n!}\exp\left(-\frac{\mathrm{d}}{\mathrm{d}y}y\frac{\mathrm{d}}{\mathrm{d}y}\right)y^n\exp(-y). \tag{48}$$

It is convenient to introduce an auxiliary function,

$$R(y,\beta) = \exp\left(-\frac{\mathrm{d}}{\mathrm{d}y}y\frac{\mathrm{d}}{\mathrm{d}y}\right)\exp(-y\beta) \tag{49}$$

and since

$$\left(-\frac{\mathrm{d}}{\mathrm{d}y}y\frac{\mathrm{d}}{\mathrm{d}y}\right)^n\exp(y) = (-1)^n n! L_n(y)\exp(y)$$

we can write

$$R(y,\beta) = \sum_{n=0}^{\infty}\beta^n L_n(\beta y)\exp(-y\beta). \tag{50}$$

Now we can express equation (48) in terms of equation (50) as

$$\begin{aligned} P_n(y) &= \frac{1}{n!}\lim_{\beta\to 1}\left(-\frac{\partial}{\partial\beta}\right)^n R(y,\beta) \\ &= \frac{1}{n!}\lim_{\beta\to 1}\left(-\frac{\partial}{\partial\beta}\right)^n\left[\exp(-y\beta)\sum_{k=0}^{\infty}\beta^k L_k(\beta y)\right] \end{aligned} \tag{51}$$

thus the auxiliary function $R(y,\beta)$ stands for the GEF of the P-distributions. Using the definition of GEF (7) we have

$$\exp(-y\beta)\sum_{k=0}^{\infty}\beta^k L_k(\beta y) = G(y,\beta,0) = \sum_{k=0}^{\infty}\beta^k L_k(y) \tag{52}$$

and substituting this result in equation (51) we obtain

$$P_n(y) = (-1)^n\sum_{k=n}^{\infty}\binom{k}{n}L_k(y). \tag{53}$$

Using equations (35) and (37) we get a direct relation between the P-distributions and the Laguerre factorial moments,

$$P_n(y) = \frac{(-1)^n}{n!}\mathcal{F}_n(y,0) \tag{54}$$

or in terms of the GFs

$$P_n(y) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\delta_+^{(k)}(y) = (1-\mathrm{d}_y)^n\delta_+(y) \tag{55}$$

which simplifies the task of calculating mean values in equation (1) when the state of the field can be written as $\hat{\rho} = \sum_{n=0}^{\infty} p_n|n\rangle\langle n|$, where p_n is the probability associated with the state $|n\rangle$. As an illustration, we display the $n = 0, 1, 2$ P-distributions,

$$P_0(y) = \delta_+(y) \qquad P_1(y) = -\delta_+^{(1)}(y) + \delta_+(y) \qquad P_2(y) = \delta_+^{(2)}(y) - 2\delta_+^{(1)}(y) + \delta_+(y).$$

5. Conclusions

We have obtained an explicit expression for the Laguerre moments and factorial moments, for ordinary and associated Laguerre polynomials, written as sums of generalized functions. We showed that the P-distribution of an electromagnetic field in a photon-number state $|n\rangle$ is proportional to the Laguerre factorial moment of order n and that it acquires a simple form when expressed as a sum of generalized functions.

Acknowledgments

SSM and DG thank the CNPq (Brasília, Brazil) for partial financial support. This work was partially supported by FAPESP (São Paulo, SP, Brazil) under contract no 2000/15084-5. We also wish to thank Professor F Castilho Alcaraz for valuable discussion and suggestions.

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